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Singular Lagrangians: some geometric structures along the Legendre map

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Abstract

New geometric structures that relate the Lagrangian and Hamiltonian formalisms defined upon a singular Lagrangian are presented. Several vector fields are constructed in velocity space that give new and precise answers to several topics such as the projectability of a vector field to a Hamiltonian vector field, the computation of the kernel of the presymplectic form of a Lagrangian formalism, the construction of the Lagrangian dynamical vector fields and the characterization of dynamical symmetries.

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1. Introduction

The dynamics associated with a first-order time-independent variational principle on a configuration manifold Q can be formulated either in its tangent bundle $\mathrm{T}Q$ (Lagrangian formalism) or in its cotangent bundle T^*Q (Hamiltonian formalism). If the variational problem is defined by the Lagrangian function L, both formulations are related through the Legendre transformation, which is given by the fibre derivative of L, $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$.

In the regular case, that is, when $\mathcal{F}L$ is a local diffeomorphism (or when the fibre Hessian is everywhere non-singular), the equivalence between both formulations is fairly simple. However, in the singular case, this correspondence between the Lagrangian and the Hamiltonian formalisms is far from trivial, and it is just this case which is the most relevant for the fundamental physical theories (as generally covariant theories, Yang–Mills theories and string theory), because the occurrence of gauge freedom is only possible within this framework. This explains the effort made since 1950 to define the Lagrangian and Hamiltonian formalisms in the singular case, to study the relations between them, their dynamics and symmetries, their quantization, and so on. In contrast to the regular case, some specific features of the singular case include constraints, arbitrary functions, gauge invariance, gauge fixing, etc.

This development has benefited from the introduction of differential-geometric methods in the study of dynamical systems—some books along this line are for instance [AM 78, Arn 89, God 69, JS 98]. A great variety of tools from differential geometry (manifolds and bundles, differential forms, metrics, connections, etc) have been widely applied since the 1970s to singular Lagrangians, achieving a fair comprehension about the Lagrangian and the Hamiltonian formalisms and their relations.

The need of fine tools in the singular case is a direct consequence of the Legendre transformation $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ being singular. For instance, if $\mathcal{F}L$ is a diffeomorphism, a Hamiltonian vector field Z in T^*Q (with respect to the canonical symplectic form ω_Q) is directly converted into a Hamiltonian vector field $Y = \mathcal{F}L^*(Z)$ in $\mathrm{T}Q$ (with respect to the symplectic form $\omega_L = \mathcal{F}L^*(\omega_Q)$, which indeed can be used to describe the Lagrangian dynamics). In the singular case, each part of this statement (which of course is not true) has to be scrutinized carefully.

The purpose of this paper is to introduce some as yet unveiled geometric structures that appear in these formalisms and that facilitate the connection between the Lagrangian and the Hamiltonian formulations in the singular case. Once the Lagrangian function is fixed, a vector field Y_h in $\mathrm{T}Q$ will be defined from an arbitrary function h in T^*Q; this is our main object. From it, once a Hamiltonian and a basis for the primary Hamiltonian constraints are chosen, another vector field Δ_h will be defined; should the Lagrangian be regular, the vector field Δ_h would be the Hamiltonian vector field of $\mathcal{F}L^*(h)$ with respect to ω_L. These constructions, and other ones related to them, provide new connections between the dynamics in both pictures. Applications include the study of the projectability of a vector field in the Lagrangian formalism to a Hamiltonian vector field, the construction of the Lagrangian dynamical vector fields, the study of the relation between the arbitrary functions of the Lagrangian and Hamiltonian dynamics, and the formulation of the dynamical symmetries (with special emphasis on the Noether symmetries); even the intrinsic construction of some structures as the kernel of the presymplectic form in tangent space will become almost trivial.

As for the geometric tools used in the paper, they are related to the fibred structure of the tangent and cotangent bundles. We use basically the fibre derivative (that is, the ordinary differentiation with respect to the fibre variables), the vertical lift (that is, the identification between points and tangent vectors in a vector space) and the canonical structures of the tangent bundle (vertical endomorphism, canonical involution) and of the cotangent bundle (the canonical differential forms).

The paper is organized as follows. Sections 2 and 3 provide some differential-geometric preliminaries concerning bundles and the fibre derivative. Section 4 contains a geometric description of Lagrangian and Hamiltonian formalisms in the singular case. The construction of the vector field Y_h is presented in section 5, together with some of its properties. Two other vector fields, R_h and Δ_h, are also presented there. Section 6 uses the mentioned constructions to study the projectability to Hamiltonian vector fields of T^*Q, and to give an explicit basis for the kernel of the presymplectic form ω_L of the Lagrangian formalism. In section 7 the preceding vector fields are used to construct the Lagrangian dynamics and to relate the arbitrary functions of Lagrangian and Hamiltonian dynamics; the dynamical symmetries of the Hamiltonian formalism are also studied in a simple way. The case of regular Lagrangians is studied in section 8. Section 9 contains a simple example. The final section is devoted to conclusions.

2. Some facts about bundles

Basic techniques concerning fibre bundles and vector bundles will be needed; in particular, the vertical vectors of a bundle and the tangent bundle of a bundle, as well as some canonical structures related to the tangent bundle. They may be found in many books, such as for instance [AM 78, AMR 88, Die 70, God 69, KMS 93, Sau 89]. In this section we recall a few of these concepts and introduce some notation.

Vertical vectors. Let $\pi\colon E \to B$ be a fibre bundle, with fibres $E_x = \pi^{-1}(x)$. The *vertical bundle* of E is the vector subbundle $\mathrm{V}(E) = \operatorname{Ker} \mathrm{T}(\pi) \subset \mathrm{T}(E)$. Its fibre at a point $e_x \in E_x$ is the tangent space to the fibre of E at x: $\mathrm{V}_{e_x}(E) = \mathrm{T}_{e_x}(E_x)$.

Let us consider a *vector bundle* $E \to B$. At each $x \in B$ we have a vector space E_x. The tangent space of E_x at a point e_x is naturally isomorphic to E_x itself, $E_x \stackrel{\cong}{\to} \mathrm{T}_{e_x}(E_x)$; this isomorphism is constructed by sending v_x to the tangent vector of the path $t \mapsto e_x + tv_x$ in E_x. Therefore, $\mathrm{T}(E_x) \cong E_x \times E_x$.

Globally this yields a canonical isomorphism $\mathrm{V}(E) \cong E \times_B E$, called the *vertical lift*

$$\begin{aligned} &E \times_B E \xrightarrow{\mathrm{vl}_E} \mathrm{V}(E) \subset \mathrm{T}(E) \\ &(e_x, v_x) \mapsto \mathrm{vl}_E(e_x, v_x) = [t \mapsto e_x + tv_x]. \end{aligned} \tag{2.1}$$

Here $E \times_B E$ denotes the fibre product (its elements are the couples $(e, e') \in E \times E$ such that $\pi(e) = \pi(e')$), considered as a vector bundle over the first factor.

The vertical lift defines a natural bijection between fibre bundle maps $E \to E$ and vertical vector fields on E: if $\xi\colon E \to E$ is a fibre bundle map, then the map

$$\xi^{\mathrm{v}}\colon E \longrightarrow \mathrm{V}(E) \subset \mathrm{T}(E) \qquad \xi^{\mathrm{v}}(e) = \mathrm{vl}_E(e, \xi(e)) \tag{2.2}$$

is a vertical vector field. This procedure applied to the identity map of E yields a canonical vertical vector field, the *Liouville vector field*, $\Delta_E(e) = \mathrm{vl}_E(e, e)$. If (x, a) are vector bundle coordinates of E—usually we will omit indices—then the local expression of Δ_E is $a^i \partial/\partial a^i$.

Some structures of $\mathrm{T}(\mathrm{T}B)$. Given a vector bundle $\pi\colon E \to B$, the tangent bundle $\mathrm{T}E$ has two vector bundle structures: $\tau_E\colon \mathrm{T}E \to E$ and $\mathrm{T}\pi\colon \mathrm{T}E \to \mathrm{T}B$. In the case of $E = \mathrm{T}B$, we obtain two different vector bundle structures over the same base. Both structures are canonically isomorphic through the *canonical involution*, $\kappa_B\colon \mathrm{T}(\mathrm{T}B) \to \mathrm{T}(\mathrm{T}B)$. Its local expression in natural coordinates is

$$\kappa(x, v; u, a) = (x, u; v, a).$$

Another map in this manifold is the *vertical endomorphism* $\mathrm{J}\colon \mathrm{T}(\mathrm{T}B) \to \mathrm{T}(\mathrm{T}B)$, whose local expression is

$$\mathrm{J}(x, v; u, a) = (x, v; 0, u).$$

Projectability. Let $\mathcal{F}\colon M \to N$ be a map between manifolds. A function $f\colon M \to \mathbb{R}$ is said to be *projectable* (through $\mathcal{F}$) if $f = \mathcal{F}^* g := g \circ \mathcal{F}$ for a certain function $g\colon N \to \mathbb{R}$. A vector field X on M is *projectable* if there exists a vector field Y on N such that $\mathrm{T}(\mathcal{F}) \circ X = Y \circ \mathcal{F}$; one also says that X and Y are $\mathcal{F}$-related. Alternatively, one has $X \cdot \mathcal{F}^*(g) = \mathcal{F}^*(Y \cdot g)$ for any function g on N.

When $\mathcal{F}$ has *constant rank*, one can use the rank theorem to obtain a characterization of the *local* projectability of a function f: this condition is that $v \cdot f = 0$ for every $v \in \operatorname{Ker} \mathrm{T}(\mathcal{F})$.

There are similar results for the local projectability of vector fields. However, let us just point out one result from the opposite side: a vector field Y on N is locally the projection of a vector field X iff Y is tangent to the image of $\mathcal{F}$.

3. Fibre derivatives

The fibre derivative will play an important role in our developments. Its definition can be found in many places (such as, for instance, [GS 73, AM 78]), since it is a relevant structure when constructing the Legendre transformation that connects Lagrangian and Hamiltonian formalisms. In a recent article [Grà 00] the fibre derivative has been studied in detail, with a view to application in singular Lagrangian dynamics. In this section we summarize some of the results of that paper.

Definition of the fibre derivative. Our framework consists of two real *vector bundles* $E \to M$ and $F \to M$ over the same base, and a *fibre* M*-bundle morphism* $f\colon E \to F$, that is, a fibre-preserving map: for each $e_x \in E_x$, $f(e_x) \in F_x$. (In [Grà 00] the more general case of E and F being affine bundles is considered; this is especially interesting, for instance, when considering higher-order or time-dependent Lagrangians, or field theory.)

The restriction of f to a fibre defines a map $f_x\colon E_x \to F_x$ between vector spaces, whose ordinary derivative at a point $e_x \in E_x$ is a linear map $\mathrm{D}f_x(e_x)\colon E_x \to F_x$. In other words, we have defined an element

$$\mathcal{F}f(e_x) := \mathrm{D}f_x(e_x) \in \mathrm{Hom}(E_x, F_x) \tag{3.1}$$

for each $e_x \in E$. Globally, this defines a fibre-preserving map

$$\mathcal{F}f\colon E \longrightarrow \mathrm{Hom}(E, F) \cong F \otimes E^* \tag{3.2}$$

which is the *fibre derivative* of f.

If the local expression of f is $(x^\mu, a^i) \mapsto (x^\mu, f^k(x, a))$, then the local expression of $\mathcal{F}f$ is

$$\mathcal{F}f(x^\mu, a^i) = \left(x^\mu, \frac{\partial f^k}{\partial a^i}(x, a)\right). \tag{3.3}$$

Since $\mathcal{F}f$ is also a fibre bundle map between vector bundles, the same procedure can be applied to compute its fibre derivative. The canonical isomorphism $\mathrm{Hom}(E, \mathrm{Hom}(E, F)) \cong \mathcal{L}^2(E; F)$ now yields the second fibre derivative, the *fibre Hessian*, which is the map

$$\mathcal{F}^2 f\colon E \longrightarrow \mathcal{L}^2(E; F) \cong \mathrm{Hom}(E \otimes E, F) \cong F \otimes E^* \otimes E^* \tag{3.4}$$

whose local expression is

$$\mathcal{F}^2 f(x^\mu, a^i) = \left(x^\mu, \frac{\partial^2 f^k}{\partial a^i\, \partial a^j}(x, a)\right). \tag{3.5}$$

This can be readily generalized to higher-order fibre derivatives.

The case of a real function. Let us note the particular case where $F = M \times \mathbb{R}$. This corresponds indeed to considering a real function $f: E \to \mathbb{R}$ on a vector bundle $\pi: E \to M$. Then its fibre derivative is a map

$$\mathcal{F}f: E \longrightarrow \mathrm{Hom}(E, M \times \mathbb{R}) =: E^* \tag{3.6}$$

of which we shall study some properties.

First, there is a close relation between the tangent map

$$\mathrm{T}(\mathcal{F}f): \mathrm{T}E \longrightarrow \mathrm{T}E^*$$

and the fibre Hessian $\mathcal{F}^2 f$ of f,

$$\mathcal{F}^2 f = \mathcal{F}(\mathcal{F}f): E \longrightarrow \mathrm{Hom}(E, E^*) \cong E^* \otimes E^*.$$

Indeed, the restriction of $\mathrm{T}_{e_x}(\mathcal{F}f)$ to vertical vectors is—thanks to the vertical lift—essentially the same map as the Hessian considered as a map $\mathcal{F}^2 f(e_x): E_x \to E_x^*$. Consequently, one has that

$$v_x \in \mathrm{Ker}\, \mathcal{F}^2 f(e_x) \qquad \Longleftrightarrow \qquad \mathrm{vl}_E(e_x, v_x) \in \mathrm{Ker}\, \mathrm{T}_{e_x}(\mathcal{F}f)$$

and since $\mathrm{Ker}\, \mathrm{T}(\mathcal{F}f) \subset \mathrm{V}(E)$, in this way we obtain the whole subbundle $\mathrm{Ker}\, \mathrm{T}(\mathcal{F}f)$. Note, in particular, that $\mathcal{F}f$ is a local diffeomorphism at $e_x \in E$ iff $\mathcal{F}^2 f(e_x)$ is a linear isomorphism.

These results can also be deduced from the local expressions of the maps; using as natural coordinates of E and E^* (x, a) and (x, α), respectively, they are:

$$\mathcal{F}f: \quad (x, a) \mapsto \left(x, \frac{\partial f}{\partial a}(x, a)\right)$$

$$\mathrm{T}(\mathcal{F}f): \quad (x, a; v, h) \mapsto \left(x, \frac{\partial f}{\partial a}(x, a); v, \frac{\partial^2 f}{\partial a\, \partial x} v + \frac{\partial^2 f}{\partial a\, \partial a} h\right)$$

$$\mathcal{F}^2 f: \quad (x, a) \mapsto \left(x, \frac{\partial^2 f}{\partial a\, \partial a}(x, a)\right).$$

Finally, we want to note the following result. If $\xi: E \to E$ is a bundle map with associated vertical field $X = \xi^{\mathrm{v}}$ on E, and $g: E \to \mathbb{R}$ is a function, then

$$X \cdot g = \langle \mathcal{F}g, \xi \rangle. \tag{3.7}$$

This can be applied, in particular, to the Liouville vector field, giving

$$(\Delta_E \cdot g)(e_x) = \langle \mathcal{F}g(e_x), e_x \rangle; \tag{3.8}$$

the fibre derivative of this expression can be computed by applying the Leibniz rule, and is

$$\mathcal{F}(\Delta_E \cdot g)(e_x) = \mathcal{F}g(e_x) + \mathcal{F}^2 g(e_x) \cdot e_x. \tag{3.9}$$

Some useful structures: Γ_h *and* Υ^g. Considering the fibre derivative $\mathcal{F}f: E \to E^*$ of f as fixed data, we are going to derive several properties of a function $h: E^* \to \mathbb{R}$ and its fibre derivatives.

We use the notation

$$\gamma_h = \mathcal{F}h \circ \mathcal{F}f: E \to E \tag{3.10}$$

for the composition $E \xrightarrow{\mathcal{F}f} E^* \xrightarrow{\mathcal{F}h} E^{**} \cong E$. Recall that this map, through the vertical lift, defines a vertical vector field γ_h^{v} on E:

$$\Gamma_h := \gamma_h^{\mathrm{v}} = \mathrm{vl}_E \circ (\mathrm{Id}_E, \mathcal{F}h \circ \mathcal{F}f): E \to E \times_M E \to \mathrm{V}E \subset \mathrm{T}E. \tag{3.11}$$

Their local expressions are

$$\gamma_h \colon (x, a) \mapsto \left(x, \frac{\partial h}{\partial \alpha}(\mathcal{F}f(x,a)) \right) \qquad \Gamma_h = (\mathcal{F}f)^* \left(\frac{\partial h}{\partial \alpha_i} \right) \frac{\partial}{\partial a^i}.$$

We can apply the chain rule to compute expressions like

$$\mathcal{F}(h \circ \mathcal{F}f) = \mathcal{F}^2 f \bullet \gamma_h \tag{3.12}$$

$$\mathcal{F}(\gamma_h) = (\mathcal{F}^2 h \circ \mathcal{F}f) \bullet \mathcal{F}^2 f. \tag{3.13}$$

Here we have, for instance, $\mathcal{F}^2 h \circ \mathcal{F}f \colon E \to E^* \to \mathrm{Hom}(E^*, E^{**}) \cong \mathrm{Hom}(E^*, E)$ and $\mathcal{F}^2 f \colon E \to \mathrm{Hom}(E, E^*)$; the symbol $\bullet$ denotes the composition between the images of both maps—it is like the contraction of vector fields with differential forms.

Note from (3.12) that if h vanishes on the image $\mathcal{F}f(E) \subset E^*$ then γ_h is in the kernel of $\mathcal{F}^2 f$. So we obtain the following result (see also [Grà 00, BGPR 86]): *suppose that $\mathcal{F}f$ has constant rank; thus, locally the image of $\mathcal{F}f$ is a submanifold of E^* that can be (locally) described by the vanishing of a set of independent functions $\phi_\mu \colon E^* \to \mathbb{R}$. Then the vectors $\gamma_{\phi_\mu}(e_x)$ are a basis for* $\mathrm{Ker}\, \mathcal{F}^2 f(e_x)$, *and the vertical vector fields Γ_{ϕ_μ} constitute a frame for* $\mathrm{Ker}\, \mathrm{T}(\mathcal{F}f)$.

As a byproduct, a function on E is (locally) projectable through $\mathcal{F}f$ to E^* iff its Lie derivative with respect to the vector fields Γ_{ϕ_μ} is zero.

Now we present a construction dual to Γ_h. Given a function $g \colon E \to \mathbb{R}$, we can use its fibre derivative $\mathcal{F}g \colon E \to E^*$ to construct a map

$$\Upsilon^g = \mathrm{vl}_{E^*} \circ (\mathcal{F}f, \mathcal{F}g) \colon E \to E^* \times_M E^* \to \mathrm{V}E^* \subset \mathrm{T}E^* \tag{3.14}$$

this is a vector field along the map $\mathcal{F}f$, with the local expression

$$\Upsilon^g = \frac{\partial g}{\partial a^i} \left(\frac{\partial}{\partial \alpha_i} \circ \mathcal{F}f \right).$$

Recall that a section of a bundle $\pi \colon E \to B$ along a map $f \colon B' \to B$ is a map $\sigma \colon B' \to E$ such that $\pi \circ \sigma = f$. In particular, a section $Z \colon B' \to \mathrm{T}B$ of $\mathrm{T}B$ along f is called a *vector field along* f; such a map derivates a function $h \colon B \to \mathbb{R}$ giving a function $Z \cdot h$ on B': $(Z \cdot h)(y) = Z(y) \cdot h$.

Note finally that, as differential operators, Γ_h and Υ^g are related by

$$\Upsilon^g \cdot h = \Gamma_h \cdot g. \tag{3.15}$$

This follows from the fact that $\Gamma_h \cdot g = \langle \mathcal{F}g, \gamma_h \rangle = \langle \mathcal{F}g, \mathcal{F}h \circ \mathcal{F}f \rangle = \Upsilon^g \cdot h$.

4. Some structures of Lagrangian and Hamiltonian formalisms

The basic concepts about singular Lagrangian and Hamiltonian formalisms—Legendre map, energy, Hamiltonian function, Hamiltonian constraints, etc—are well known and can be found in several papers, such as for instance [BGPR 86, BK 86, Car 90, GNH 78, MMS 83, MT 78]. Now we will recall some of these concepts, also introducing some recent results from [Grà 00].

Connection between the Lagrangian and the Hamiltonian spaces. Let us consider a first-order autonomous Lagrangian on a configuration space Q, that is to say, a map $L\colon \mathrm{T}Q \to \mathbb{R}$. Its fibre derivative (Legendre transformation) and fibre Hessian are maps

$$\mathcal{F}L\colon \mathrm{T}Q \longrightarrow \mathrm{T}^*Q$$

$$\mathcal{F}^2L = \mathcal{F}(\mathcal{F}L)\colon \mathrm{T}Q \longrightarrow \mathrm{Hom}(\mathrm{T}Q, \mathrm{T}^*Q) = \mathrm{T}^*Q \otimes \mathrm{T}^*Q.$$

The local expression of $\mathcal{F}L$ is $\mathcal{F}L(q, \dot{q}) = (q, \hat{p})$, where

$$\hat{p} = \frac{\partial L}{\partial \dot{q}}$$

are the momenta. If the Legendre map is a local diffeomorphism—equivalently the Hessian is everywhere non-singular—the Lagrangian L is called *regular*, otherwise it is called *singular*—this is our focus of interest.

We assume that the Legendre transformation of L has connected fibres and is a submersion onto a closed submanifold $P_o \subset \mathrm{T}^*Q$, the *primary Hamiltonian constraint submanifold*—that is to say, L is an almost regular Lagrangian in the terminology of [GN 79]. This is the most basic technical requirement to develop a Hamiltonian formulation from a singular Lagrangian L, though from a local viewpoint it suffices to have $\mathcal{F}L$ of constant rank. Locally P_o can be described by the vanishing of an independent set of functions ϕ_μ, called the *primary Hamiltonian constraints*. According to the preceding section, the vectors $\gamma_\mu = \gamma_{\phi_\mu}$ constitute a basis for the kernel of $\mathcal{F}^2L$, and the vertical fields $\Gamma_\mu = \Gamma_{\phi_\mu}$ constitute a frame for $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$.

The *energy* of L is defined by

$$E_L = \Delta_{\mathrm{T}Q} \cdot L - L.$$

Due to the properties of the Liouville vector field (3.8), (3.9),

$$E_L(u_q) = \langle \mathcal{F}L(u_q), u_q \rangle - L(u_q) \tag{4.1}$$

$$\mathcal{F}E_L(u_q) = \mathcal{F}^2L(u_q) \cdot u_q. \tag{4.2}$$

This shows at once that $\Gamma_\mu \cdot E_L = \langle \mathcal{F}E_L, \gamma_\mu \rangle = 0$, that is to say, the energy is projectable (through $\mathcal{F}L$) to a function $H\colon \mathrm{T}^*Q \to \mathbb{R}$ called a *Hamiltonian*,

$$E_L = H \circ \mathcal{F}L$$

which is unique on the primary Hamiltonian constraint submanifold.

A resolution of the identity. Given an almost regular Lagrangian L, the choice of a Hamiltonian and set of primary Hamiltonian constraints yields a (local) resolution of the identity map of $\mathrm{T}Q$ as follows.

There exist functions v^μ (defined on an open set of $\mathrm{T}Q$) such that, locally,

$$\mathrm{Id}_{\mathrm{T}Q} = \gamma_H + \sum_\mu \gamma_\mu\, v^\mu. \tag{4.3}$$

Moreover,

$$\mathcal{I}d_{\mathrm{Hom}(\mathrm{T}Q,\mathrm{T}Q)} = M \bullet \mathcal{F}^2L + \sum_\mu \gamma_\mu \otimes \mathcal{F}v^\mu \tag{4.4}$$

where

$$M = (\mathcal{F}^2H \circ \mathcal{F}L) + \sum_\mu (\mathcal{F}^2\phi_\mu \circ \mathcal{F}L)\, v^\mu. \tag{4.5}$$

(Note that $\mathcal{F}^2L$ is a map $\mathrm{T}Q \to \mathrm{Hom}(\mathrm{T}Q, \mathrm{T}^*Q)$ and M is a map $\mathrm{T}Q \to \mathrm{Hom}(\mathrm{T}^*Q, \mathrm{T}Q) = \mathrm{T}Q \otimes \mathrm{T}Q$.)

Since the functions υ^μ and their properties will be instrumental throughout the paper, we will recall the proof of this result [Grà 00]. Application of the chain rule (3.12) to the definition of H yields $\mathcal{F}E_L(u_q) = \mathcal{F}^2L(u_q) \cdot \gamma_H(u_q)$, and so using (4.2) we obtain

$$\mathcal{F}^2L(u_q) \cdot (u_q - \gamma_H(u_q)) = 0.$$

The terms in parentheses are in $\mathrm{Ker}\,\mathcal{F}^2L(u_q)$, thus there exist numbers $\upsilon^\mu(u_q)$ such that $u_q - \gamma_H(u_q) = \sum_\mu \gamma_\mu(u_q)\,\upsilon^\mu(u_q)$, which is equation (4.3). Finally, using (3.13) and the Leibniz rule, one can compute the fibre derivative of (4.3); the result is equation (4.4).

The above results can be given a slightly different form, using the identification of bundle maps $\mathrm{T}Q \to \mathrm{T}Q$ with vertical vector fields on $\mathrm{T}Q$. For instance, equation (4.3) can be rewritten as

$$\Delta_{\mathrm{T}Q} = \Gamma_H + \sum_\mu \upsilon^\mu\,\Gamma_\mu. \tag{4.6}$$

Note that application of (4.4) to γ_ν yields $\gamma_\nu = \sum_\mu \gamma_\mu \langle \mathcal{F}\upsilon^\mu, \gamma_\nu \rangle$. So we have

$$\Gamma_\nu \cdot \upsilon^\mu = \langle \mathcal{F}\upsilon^\mu, \gamma_\nu \rangle = \delta^\mu_\nu \tag{4.7}$$

where we have applied equation (3.7). This shows that the functions υ^μ are not projectable; in a certain sense, they correspond to the velocities that cannot be retrieved from the momenta through the Legendre map.

Let us finally note that the local expressions of equations (4.4) and (4.5) were initially deduced in [BGPR 86] by derivating the local expression of (4.3), which is

$$\dot{q}^i = \mathcal{F}L^*\left(\frac{\partial H}{\partial p_i}\right) + \sum_\mu \mathcal{F}L^*\left(\frac{\partial \phi_\mu}{\partial p_i}\right)\upsilon^\mu.$$

The Euler–Lagrange equation. So far we have not considered the equations of motion. We will deal with them in several forms.

Let ω_Q be the canonical 2-form of T^*Q (in coordinates $\mathrm{d}q^i \wedge \mathrm{d}p_i$). One defines the presymplectic form in $\mathrm{T}Q$

$$\omega_L = \mathcal{F}L^*(\omega_Q).$$

It is a symplectic form iff the Lagrangian is regular. Then a path $\gamma\colon I \to Q$ is a solution of the Euler–Lagrange equation iff

$$i_{\ddot{\gamma}}\omega_L = \mathrm{d}E_L \circ \dot{\gamma}. \tag{4.8}$$

A second representation of the equation of motion is

$$\mathcal{E}_L \circ \ddot{\gamma} = 0 \tag{4.9}$$

where $\mathcal{E}_L\colon \mathrm{T}^2Q \to \mathrm{T}^*Q$ is the *Euler–Lagrange form* of L (see for instance [CLM 91, Tul 75]); T^2Q denotes the second-order tangent bundle of Q. $\mathcal{E}_L$ is a 1-form along the projection $\mathrm{T}^2Q \to Q$, with local expression

$$\mathcal{E}_L = [L]_i\,\mathrm{d}q^i \qquad [L]_i = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right). \tag{4.10}$$

A third version of the Euler–Lagrange equation can be written using the time-evolution operator K that connects Lagrangian and Hamiltonian formalisms. This operator was expressed in [GP 89] as a vector field along $\mathcal{F}L$ satisfying certain properties that determine it completely. The local expression of K is

$$K(q, \dot{q}) = \left(q, \widehat{p}; \dot{q}, \frac{\partial L}{\partial q} \right).$$

In coordinates, K was first introduced [BGPR 86] as a differential operator (see also [CL 87, Car 90]). Then its local expression reads

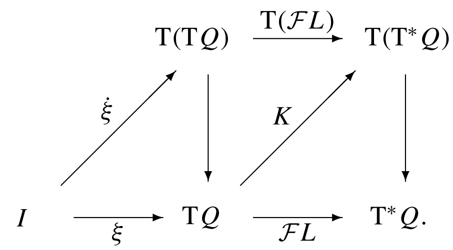
$$K \cdot h = \mathcal{F}L^* \left(\frac{\partial h}{\partial q} \right) \dot{q} + \mathcal{F}L^* \left(\frac{\partial h}{\partial p} \right) \frac{\partial L}{\partial q}. \tag{4.11}$$

(In a time-dependent framework it would hold an additional piece, $\mathcal{F}L^*(\partial h/\partial t)$.) The operator K is a useful tool in the theory of singular Lagrangians: it can be used (see below) to express the equations of motion [GP 89], to relate the Lagrangian and the Hamiltonian constraints [BGPR 86, CL 87, Pon 88], to study the symmetries of the equations of motion [GP 88, BGGP 89, FP 90, GP 92b, GP 94, GP 00] and, more recently, to study Lagrangian systems with generic singularities [PV 00]. See also [GPR 91, GP 95].

Using this operator, a path $\xi: I \to \mathrm{T}Q$ is the lift $\dot{\gamma}$ of a solution of the Euler–Lagrange equation iff

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi. \tag{4.12}$$

The following diagram shows all the objects involved:



The Hamilton–Dirac equation. In the singular case, Hamiltonian dynamics was first studied by Dirac and Bergmann [Dir 50, AB 51, Dir 64]. A path $\eta: I \to P_o$ is a solution of the Hamilton–Dirac equation if there exist functions λ^μ such that

$$\dot{\eta} = Z_H \circ \eta + \sum_\mu \lambda^\mu Z_\mu \circ \eta. \tag{4.13}$$

Here we denote by Z_h the Hamiltonian vector field defined by h: it satisfies

$$i_{Z_h} \omega_Q = \mathrm{d}h$$

and, as a differential operator, it is related to the Poisson bracket by

$$Z_h = \{-, h\}.$$

We have also put $Z_\mu = Z_{\phi_\mu}$.

Another geometric version of Dirac's theory can be obtained by considering $j: P_o \hookrightarrow \mathrm{T}^*Q$ and the presymplectic form $\omega_o = j^*(\omega_Q)$. Then the Hamilton–Dirac equation for a path $\eta: I \to P_o$ is

$$i_{\dot{\eta}} \omega_o = \mathrm{d}H_o \circ \eta \tag{4.14}$$

where H_o is the Hamiltonian on P_o [GNH 78, BK 86].

Using the operator K, the Hamilton–Dirac equation can also be written as

$$\dot{\eta} = K \circ \mathrm{T}(\tau_Q^*) \circ \dot{\eta} \tag{4.15}$$

for a path η in T^*Q [GP 89]—see also [BGPR 86, Tul 76].

Of course, the Hamiltonian dynamics is defined so as to be equivalent to the Lagrangian dynamics, in the sense that if $\xi: I \to \mathrm{T}Q$ is a solution of the Euler–Lagrange equation then $\eta: I \to \mathrm{T}^*Q$ defined as $\eta = \mathcal{F}L \circ \xi$ satisfies the Hamilton–Dirac equation, and conversely taking η and defining $\xi = (\tau_Q^* \circ \eta)^{\cdot}$ from it. We will say that such ξ, η are a couple of related solutions.

Some further relations involving the operator K. Since the same dynamics is written in different ways, there are relations between the different structures involved. Let us point out first

$$K \cdot h = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}L^*(h) + \langle \mathcal{E}_L, \gamma_h \rangle. \tag{4.16}$$

Here there is an abuse of notation that requires some explanation. On the right-hand side we have a function $\mathcal{F}L^*(h)$ on $\mathrm{T}Q$, whose total time derivative (see, for instance, [Sau 89, CLM 91]) is a function on T^2Q, and the contraction of $\mathcal{E}_L$ with γ_h, considered as a function on T^2Q; however, the sum of both functions turns out to not depend on the acceleration, so it is a function on $\mathrm{T}Q$, just as for the left-hand side.

The local expression of (4.16) first appeared in [GP 92b].

Though for singular Lagrangians the Lagrangian and the Hamiltonian dynamics are not, in general, completely determined, equation (4.16) shows that, when considering solutions of Euler–Lagrange and Hamilton–Dirac equations, the evolution operator K gives an unambiguous time derivative of a function in Hamiltonian space expressed in Lagrangian terms. In particular, taking $h = \phi_\mu$, we obtain the *primary Lagrangian constraints*

$$\chi_\mu := K \cdot \phi_\mu = \langle \mathcal{E}_L, \gamma_\mu \rangle : \mathrm{T}Q \to \mathbb{R} \tag{4.17}$$

note that they also arise directly from (4.9) as a consistency condition—this is due to the fact that γ_μ are in the kernel of $\mathcal{F}^2L$. The vanishing of the primary Lagrangian constraints defines the *primary Lagrangian subset* $V_1 \subset \mathrm{T}Q$, which we will assume to be a submanifold. Note that the functions χ_μ are not necessarily independent, and indeed may vanish identically.

Now we can relate the operator K with the Hamiltonian evolution. A very important result for our purposes is that

$$K \cdot h = \mathcal{F}L^*\{h, H\} + \sum_\mu \mathcal{F}L^*\{h, \phi_\mu\}\, v^\mu \tag{4.18}$$

where the functions of equation (4.3) appear again. The proof can be found in [BGPR 86], and in [GPR 91] for higher-order Lagrangians. This result can also be expressed as an equality between maps (in this case, vector fields along $\mathcal{F}L$) rather than as an equality of differential operators:

$$K = Z_H \circ \mathcal{F}L + \sum_\mu v^\mu\, (Z_\mu \circ \mathcal{F}L). \tag{4.19}$$

An immediate consequence of (4.18) is

$$\Gamma_\mu \cdot (K \cdot h) = \mathcal{F}L^*\{h, \phi_\mu\}. \tag{4.20}$$

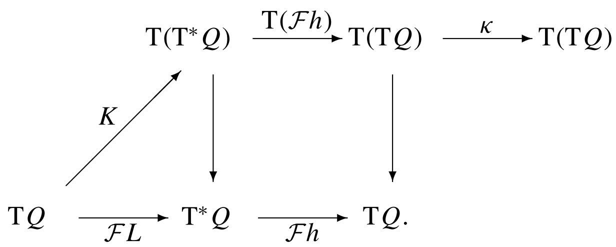
This provides us with a test of projectability: the function $K \cdot h$ is projectable iff h is a first-class function (with respect to P_o). Recall that a function $h\colon \mathrm{T}^*Q \to \mathbb{R}$ is said to be *first class* with respect to a submanifold $P \subset \mathrm{T}^*Q$ if the Hamiltonian vector field Z_h is tangent to P, which means that $\{h, \phi\} \underset{P}{\approx} 0$ for any constraint ϕ defining the submanifold. (The notation $f \underset{M}{\approx} 0$ means that $f(x) = 0$ for all $x \in M$ (Dirac's weak equality); for instance $\phi_\mu \underset{P_o}{\approx} 0$ and $\chi_\mu \underset{V_1}{\approx} 0$.)

5. Some canonical vector fields

The vector field Y_h. Let $h\colon \mathrm{T}^*Q \to \mathbb{R}$ be a function in phase space. Its fibre derivative is a map $\mathcal{F}h\colon \mathrm{T}^*Q \to \mathrm{T}Q$, so we can define another map

$$Y_h := \kappa \circ \mathrm{T}(\mathcal{F}h) \circ K \tag{5.1}$$

where K is the time-evolution operator of L and $\kappa\colon \mathrm{T}(\mathrm{T}Q) \to \mathrm{T}(\mathrm{T}Q)$ is the canonical involution of $\mathrm{T}(\mathrm{T}Q)$. Let us show all this in a diagram:



Using the local expressions of all the objects involved, one obtains the local expression of Y_h:

$$Y_h(q, \dot q) = \left(q, \dot q;\ \frac{\partial h}{\partial p}(\mathcal{F}L(q,\dot q)), \dot q \frac{\partial^2 h}{\partial q\, \partial p}(\mathcal{F}L(q,\dot q)) + \frac{\partial L}{\partial q}\frac{\partial^2 h}{\partial p\, \partial p}(\mathcal{F}L(q,\dot q)) \right). \tag{5.2}$$

Proposition 1. *The map Y_h is a vector field on* $\mathrm{T}Q$*, with local expression*

$$Y_h = \mathcal{F}L^*\{q, h\}\, \frac{\partial}{\partial q} + K \cdot \{q, h\}\, \frac{\partial}{\partial \dot q}. \tag{5.3}$$

It has the following properties:

$$\mathrm{J} \circ Y_h = \Gamma_h \tag{5.4}$$

$$Y_g \cdot (\mathcal{F}L^* h) = \mathcal{F}L^*\{h, g\} + \Gamma_h \cdot (K \cdot g) \tag{5.5}$$

$$Y_g \cdot (K \cdot h) = K \cdot \{h, g\} + Y_h \cdot (K \cdot g) \tag{5.6}$$

$$\mathrm{T}(\mathcal{F}L) \circ Y_g = Z_g \circ \mathcal{F}L + \Upsilon^{K \cdot g}. \tag{5.7}$$

Proof. The fact that Y_h is a vector field is a direct consequence of its local expression (5.2). It also follows from

$$\tau_{\mathrm{T}Q} \circ Y_h = \tau_{\mathrm{T}Q} \circ \kappa \circ \mathrm{T}(\mathcal{F}h) \circ K = \mathrm{T}(\tau_Q) \circ \mathrm{T}(\mathcal{F}h) \circ K = \mathrm{T}(\tau_Q^*) \circ K = \mathrm{Id}_{\mathrm{T}Q}\,.$$

The alternative (and more suggestive) local expression (5.3) of Y_h is also clear from (5.2), as well as the fact that $\mathrm{J} \circ Y_h = \Gamma_h$, where J is the vertical endomorphism of $\mathrm{T}(\mathrm{T}Q)$.

The following two equations can be proved from their local expressions. This is simpler for the first one, (5.5): its left- and right-hand sides read in coordinates

$$\left(\widehat{\frac{\partial h}{\partial q}} + \widehat{\frac{\partial h}{\partial p}}\frac{\partial^2 L}{\partial \dot{q}\,\partial q}\right)\widehat{\frac{\partial g}{\partial p}} + \widehat{\frac{\partial h}{\partial p}}\frac{\partial^2 L}{\partial \dot{q}\,\partial \dot{q}}\left(\widehat{\frac{\partial^2 g}{\partial p\,\partial q}}\dot{q} + \widehat{\frac{\partial^2 g}{\partial p\,\partial p}}\frac{\partial L}{\partial q}\right)$$

(we have put $\widehat{h} = \mathcal{F}L^* h$ to simplify the notation).

Regarding the second equation, (5.6), one has to prove $Y_g \cdot (K \cdot h) - Y_h \cdot (K \cdot g) = K \cdot \{h, g\}$. The terms remaining after the antisymmetrization of $Y_g(K \cdot h)$ with respect to (g, h) can be arranged to read

$$\left(\dot{q}\;\mathcal{F}L^*\;\frac{\partial}{\partial q} + \frac{\partial L}{\partial q}\;\mathcal{F}L^*\;\frac{\partial}{\partial p}\right)\left(\frac{\partial h}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial h}{\partial p}\frac{\partial g}{\partial q}\right)$$

which is $K \cdot \{h, g\}$.

Finally, equation (5.7) is obtained by using relation (3.15) to express equation (5.5) as an equality between vector fields along $\mathcal{F}L$. □

The vector fields R_h and Δ_h. Equation (5.7) shows explicitly an obstruction for the projectability of Y_g to the Hamiltonian vector field Z_g. In the discussion of this issue it will be interesting to consider the vertical vector field

$$R_h = \Gamma_{\{h,H\}} + v^\mu \Gamma_{\{h,\phi_\mu\}} \tag{5.8}$$

defined from any function h on phase space—from now on we use the summation convention for the Greek indices associated with the primary constraints. Note that R_h depends on the choice of the Hamiltonian H and the primary Hamiltonian constraints ϕ_μ. The action of R_h on projectable functions is

$$R_g \cdot \mathcal{F}L^* h = \Gamma_h \cdot (K \cdot g) - \mathcal{F}L^*\{g, \phi_\mu\}\,\Gamma_h \cdot v^\mu \tag{5.9}$$

which is a kind of generalization of (4.20). To prove it, first we apply R_g to $\mathcal{F}L^* h$, then we use the symmetry property

$$\Gamma_h \cdot \mathcal{F}L^*(g) = \mathcal{F}^2 L(\gamma_g, \gamma_h) = \Gamma_g \cdot \mathcal{F}L^*(h) \tag{5.10}$$

and finally we apply equation (4.18) to let K appear explicitly.

The interest of the vector field R_h comes from the fact that it appears when taking equation (5.6) and rewriting it using relations (4.18) and (5.5); after some cancellations one arrives at

$$R_h \cdot (K \cdot g) + \mathcal{F}L^*\{h, \phi_\mu\}\,Y_g \cdot v^\mu = R_g \cdot (K \cdot h) + \mathcal{F}L^*\{g, \phi_\mu\}\,Y_h \cdot v^\mu. \tag{5.11}$$

In other words, the left-hand side is symmetric in (g, h). We can develop this further, applying equation (4.18) again to make K disappear from (5.11). A convenient organization of the terms, together with some additional cancellations due to the symmetry property (5.10), finally yields another symmetric equation:

$$\mathcal{F}L^*\{h, \phi_\mu\}\,(Y_g - R_g) \cdot v^\mu = \mathcal{F}L^*\{g, \phi_\mu\}\,(Y_h - R_h) \cdot v^\mu. \tag{5.12}$$

This suggests to define, for any function g in phase space, the vector field

$$\Delta_g = Y_g - R_g. \tag{5.13}$$

Proposition 2. *The vector field Δ_g has the following properties:*

$$\mathrm{J} \circ \Delta_g = \Gamma_g \tag{5.14}$$

$$\Delta_g \cdot v^\mu = -\mathcal{F}L^*\{g, \phi_\nu\}\, M(\mathcal{F}v^\mu, \mathcal{F}v^\nu) \tag{5.15}$$

$$\Delta_g \cdot (\mathcal{F}L^* h) = \mathcal{F}L^*\{h, g\} + \mathcal{F}L^*\{g, \phi_\mu\}\, \Gamma_h \cdot v^\mu \tag{5.16}$$

$$\mathrm{T}(\mathcal{F}L) \circ \Delta_g = Z_g \circ \mathcal{F}L + \mathcal{F}L^*\{g, \phi_\mu\}\, \Upsilon^{v^\mu}. \tag{5.17}$$

Proof. The first property is a consequence of the same property of Y_g and the fact that R_g is vertical.

The second property gives the action of Δ_g on the non-projectable functions v^μ. To prove it, we consider equation (5.12),

$$\mathcal{F}L^*\{h, \phi_\mu\}\ \Delta_g \cdot v^\mu = \mathcal{F}L^*\{g, \phi_\mu\}\ \Delta_h \cdot v^\mu;$$

taking for h the configuration variables $h = q^i$, one obtains

$$(\Delta_g \cdot v^\mu)\, \gamma_\mu = -\mathcal{F}L^*\{g, \phi_\mu\}\, M \bullet \mathcal{F}v^\mu$$

with $M\colon \mathrm{T}Q \to \mathrm{Hom}(\mathrm{T}^*Q, \mathrm{T}Q)$ given by equation (4.5). Then contraction with $\mathcal{F}v^\nu$ and use of the property (4.7) finally yields equation (5.15).

Subtracting equations (5.5) and (5.9) yields (5.16).

Finally, using the relation (3.15) we can remove the function h from the preceding equation to obtain an equality between vector fields along $\mathcal{F}L$, thus obtaining (5.17). □

Some additional properties. The vector field on $\mathrm{T}Q$ Γ_h and the vector field along $\mathcal{F}L$ Υ^f are defined in terms of the fibre derivative, and a trivial application of Leibniz's rule shows that

$$\Gamma_{h_1 h_2} = \mathcal{F}L^*(h_1)\Gamma_{h_2} + \mathcal{F}L^*(h_2)\Gamma_{h_1} \tag{5.18}$$

$$\Upsilon^{f_1 f_2} = f_1 \Upsilon^{f_2} + f_2 \Upsilon^{f_1}. \tag{5.19}$$

Similarly, one can compute

$$Y_{h_1 h_2} = \mathcal{F}L^*(h_1)Y_{h_2} + \mathcal{F}L^*(h_2)Y_{h_1} + (K \cdot h_1)\Gamma_{h_2} + (K \cdot h_2)\Gamma_{h_1} \tag{5.20}$$

$$R_{h_1 h_2} = \mathcal{F}L^*(h_1)R_{h_2} + \mathcal{F}L^*(h_2)R_{h_1} + (K \cdot h_1)\Gamma_{h_2} + (K \cdot h_2)\Gamma_{h_1} \tag{5.21}$$

$$\Delta_{h_1 h_2} = \mathcal{F}L^*(h_1)\Delta_{h_2} + \mathcal{F}L^*(h_2)\Delta_{h_1}. \tag{5.22}$$

The last equation, which is obtained immediately by subtracting the two previous ones, shows that the vector field Δ_h is also a first-order differential operator on h.

6. Applications to the kinematics

The projectability to a Hamiltonian vector field. In equations (5.15)–(5.17) there is a common term $\mathcal{F}L^*\{g, \phi_\mu\}$ whose vanishing gives an answer to the question of projectability.

Theorem 1. *Let L be an almost regular Lagrangian. The necessary and sufficient condition for the Hamiltonian vector field Z_g in T^*Q to be the projection (through the Legendre transformation) of a vector field in $\mathrm{T}Q$ is that g should be a first-class function with respect to the primary Hamiltonian constraint submanifold $P_o \subset \mathrm{T}^*Q$.*

Then the vector field Δ_g projects to Z_g:

$$\mathrm{T}(\mathcal{F}L) \circ \Delta_g = Z_g \circ \mathcal{F}L. \tag{6.1}$$

Any other vector field projecting to Z_g is obtained by adding to Δ_g any vector field in the kernel of the tangent map $\mathrm{T}(\mathcal{F}L)$.

Proof. As we have said in section 2, the condition for a vector field in T^*Q to be a projection is its tangency to $P_o = \mathcal{F}L(\mathrm{T}Q)$. When this vector field is the Hamiltonian vector field Z_g this means that g is a first-class function with respect to the primary constraint submanifold P_o, that is, $\mathcal{F}L^*\{g, \phi_\mu\} = 0$. Then (5.17) shows that Δ_g projects to Z_g.

The last assertion is obvious, since the vector fields that project to zero are those in $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$. □

Comparing (5.17) and (5.7) one realizes that the appropriate vector field candidate to project to Z_g is Δ_g. This is because the condition that $\Upsilon^{K\cdot g} = 0$, which is equivalent to $\mathcal{F}(K \cdot g) = 0$, is more restrictive than g being first class. Indeed, $\mathcal{F}(K \cdot g) = 0$ means that any vertical vector field acting on $K \cdot g$ yields zero, then in particular $\Gamma_\mu \cdot (K \cdot g) = \mathcal{F}L^*\{g, \phi_\mu\} = 0$ by (4.20). Of course, when $\mathcal{F}(K \cdot g) = 0$ we can also say that Y_g projects to Z_g. This is also a consequence of the fact that if $\mathcal{F}(K \cdot g) = 0$ then R_g is in $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$.

Equation (6.1) in the theorem is a direct consequence of equation (5.17) in proposition 2 when g is first class. Let us rewrite equations (5.15) and (5.16) accordingly:

Proposition 3. *Let $g: \mathrm{T}^*Q \to \mathbb{R}$ be a first-class function with respect to the primary Hamiltonian constraint submanifold $P_o \subset \mathrm{T}^*Q$. Then the following results hold:*

$$\Delta_g \cdot v^\mu = 0 \tag{6.2}$$

$$\Delta_g \cdot \mathcal{F}L^* h = \mathcal{F}L^*\{h, g\} \text{ for any function } h. \tag{6.3}$$

Recalling (4.7), $\Gamma_\nu \cdot v^\mu = \delta^\mu_\nu$, note that equation (6.2) singles out Δ_g, among the set of vector fields projecting to Z_g, as the only one whose action on the non-projectable functions v^μ is zero.

Now let us study some commutators among vector fields.

Proposition 4. *Let $\phi, \phi': \mathrm{T}^*Q \to \mathbb{R}$ be primary Hamiltonian constraints, and $g, g': \mathrm{T}^*Q \to \mathbb{R}$ be first-class functions with respect to the primary Hamiltonian constraint submanifold $P_o \subset \mathrm{T}^*Q$. Then the following results hold:*

$$[\Gamma_\phi, \Gamma_{\phi'}] = 0 \tag{6.4}$$

$$[\Delta_g, \Delta_{g'}] = -\Delta_{\{g,g'\}} \tag{6.5}$$

$$[\Delta_g, \Gamma_\phi] = -\Gamma_{\{g,\phi\}} - [R_g - \Gamma_{\{g,H\}}, \Gamma_\phi]. \tag{6.6}$$

Proof. The first result is well known, we include it for the sake of completeness, and it is readily proved in coordinates taking into account that $\Gamma_\phi \cdot \mathcal{F}L^*(h) = 0$ for any function h.

For the second result, to show the equality of both vector fields it is enough to prove that both coincide as differential operators when acting on projectable functions (this is a consequence of equation (6.3), together with $[Z_g, Z_{g'}] = Z_{\{g',g\}}$) and on the non-projectable functions v^μ (this is a trivial consequence of equation (6.2)).

One can proceed in the same way to prove the third commutator. To this end, we first prove that

$$[\Delta_g, \Gamma_\mu] = 0. \tag{6.7}$$

On projectable functions the Lie bracket of the vector fields is zero; this is due to equation (6.3), and the fact that Γ_μ applied to any projectable function gives zero. On the non-projectable functions v^μ, equation (6.2) and the fact that $\Gamma_\mu \cdot v^\nu$ is constant also yields zero.

Now let us deal with the general case. First, locally we can express $\phi = a^\mu \phi_\mu$ for some functions a^μ. Then

$$\Gamma_{a^\mu \phi_\mu} = \mathcal{F}L^*(a^\mu)\Gamma_\mu$$

and $[\Delta_g, \Gamma_\phi] = [\Delta_g, \mathcal{F}L^*(a^\mu)\Gamma_\mu] = \Delta_g \cdot \mathcal{F}L^*(a^\mu)\, \Gamma_\mu$, thanks to (6.7). Using (6.3) we obtain

$$[\Delta_g, \Gamma_\phi] = \mathcal{F}L^*\{a^\mu, g\}\, \Gamma_\mu.$$

Considering $\{g, \phi\}$ we have $\Gamma_{\{g,\phi\}} = \mathcal{F}L^*(a^\mu)\Gamma_{\{g,\phi_\mu\}} + \mathcal{F}L^*\{g, a^\mu\}\Gamma_\mu$, and so we obtain

$$[\Delta_g, \Gamma_\phi] + \Gamma_{\{g,\phi\}} = \mathcal{F}L^*(a^\mu)\Gamma_{\{g,\phi_\mu\}}.$$

Finally, $\Gamma_\phi \cdot v^\mu = \mathcal{F}L^*(a^\mu)$, so we arrive at

$$[\Delta_g, \Gamma_\phi] + \Gamma_{\{g,\phi\}} = (\Gamma_\phi \cdot v^\mu)\, \Gamma_{\{g,\phi_\mu\}}. \tag{6.8}$$

To obtain (6.6), note that by definition $R_g - \Gamma_{\{g,H\}} = v^\mu \Gamma_{\{g,\phi_\mu\}}$, and since by (6.4) the Γs of constraints commute, $[R_g - \Gamma_{\{g,H\}}, \Gamma_\phi] = [v^\mu \Gamma_{\{g,\phi_\mu\}}, \Gamma_\phi] = -(\Gamma_\phi \cdot v^\mu)\, \Gamma_{\{g,\phi_\mu\}}$. □

Note, moreover, that using the relation between Y_g and Δ_g we can rewrite equation (6.6) as

$$[\Delta_g + v^\mu \Gamma_{\{g,\phi_\mu\}}, \Gamma_\phi] = \Gamma_{\{\phi,g\}} = [Y_g - \Gamma_{\{g,H\}}, \Gamma_\phi]. \tag{6.9}$$

The kernel of the presymplectic form in TQ. Here we will show that the vector fields Δ_g provide an easy explicit construction of the kernel of the presymplectic form $\omega_L = \mathcal{F}L^*\omega_Q$ of the Lagrangian formalism.

If a vector field Y in TQ projects through $\mathcal{F}L$ to a vector field Z in T$^*Q$, we have

$$i_Y\, \omega_L = \mathcal{F}L^*\left(i_Z\, \omega_Q\right).$$

This shows trivially that $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L) \subset \mathrm{Ker}\,\omega_L$—indeed it is a well known fact that $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L) = \mathrm{Ker}\,\omega_L \cap \mathrm{V}(\mathrm{T}Q)$. So the vector fields Γ_μ are part of a basis for $\mathrm{Ker}\,\omega_L$.

Now let us assume that the matrix of Poisson's brackets $\{\phi_\mu, \phi_\nu\}$ has constant rank. Then one can find an appropriate set (ϕ_μ) of independent primary Hamiltonian constraints which are split into first class ϕ_{μ_o}—their Poisson bracket with any primary Hamiltonian constraint vanishes on P_o—and second class $\phi_{\mu_o'}$—see among others [DLGP 84]. As the functions ϕ_{μ_o} are first class, the corresponding vector field $\Delta_{\mu_o} = \Delta_{\phi_{\mu_o}}$ projects to the Hamiltonian vector field Z_{μ_o}, and since

$$i_{\Delta_{\mu_o}}\omega_L = \mathcal{F}L^*\left(i_{Z_{\mu_o}}\omega_Q\right) = \mathcal{F}L^*(\mathrm{d}\phi_{\mu_o}) = \mathrm{d}\mathcal{F}L^*(\phi_{\mu_o}) = 0$$

we conclude that Δ_{μ_o} is also in $\mathrm{Ker}\,\omega_L$.

Note that the vector fields Δ_μ are linearly independent, since application of the vertical endomorphism yields independent vector fields, $\mathrm{J} \circ \Delta_\mu = \Gamma_\mu$; moreover, they are also independent of Γ_μ. Finally, the dimension of $\mathrm{Ker}\,\omega_L$ and the number of primary Hamiltonian constraints plus the number of first-class ones coincide (see for instance [MMS 83]). So we have proved the following result:

Theorem 2. *The kernel of ω_L has a basis constituted by the vector fields Γ_μ, associated with the primary Hamiltonian constraints ϕ_μ, and the vector fields Δ_{μ_o}, associated with a basis of the first-class primary Hamiltonian constraints ϕ_{μ_o}.*

This kernel has been studied in the literature on singular Lagrangians for its interest in the classification of the constraints [CLR 88, Car 90, MR 92]. An explicit computation of the kernel was first presented in [PSS 99] (see equations (2.13*a*) and (2.13*b*) of that paper), but in a coordinate, rather than a geometric framework. In that paper the kernel was given in a slightly different basis, for Δ_{μ_o} in that paper is the present Δ_{μ_o} except for the term $v^\nu \Gamma_{\{\phi_{\mu_o},\phi_\nu\}}$, which is a combination of the vector fields Γ_μ, also in the kernel. The present basis is preferable because it gives the commutation relations in their simplest form. Indeed, if

$$\{\phi_{\mu_o}, \phi_{\nu_o}\} = B^{\rho_o}_{\mu_o\nu_o}\phi_{\rho_o} + \mathrm{O}(\phi^2)$$

(the Poisson bracket of first-class constraints is first class), then, taking into account proposition 4, the algebra reads

$$\begin{aligned} &[\Gamma_\mu, \Gamma_\nu] = 0 \\ &[\Gamma_\mu, \Delta_{\nu_o}] = 0 \\ &[\Delta_{\mu_o}, \Delta_{\nu_o}] = \mathcal{F}L^*(B^{\rho_o}_{\nu_o\mu_o})\,\Delta_{\rho_o}. \end{aligned} \tag{6.10}$$

7. Applications to dynamics and symmetries

Lagrangian dynamics. Here we will give an explicit expression of the Lagrangian dynamics in terms of vector fields. Though in the case of a singular Lagrangian the Euler–Lagrange equation cannot be written in normal form, one can try to express its solutions in terms of integral curves of some dynamical vector fields. For instance, consider the Euler–Lagrange equation in the form (4.12): $\mathrm{T}(\mathcal{F}L)\circ\dot{\xi} = K\circ\xi$. Let $V \subset \mathrm{T}Q$ be a submanifold and X^{L} a second-order vector field in $\mathrm{T}Q$ *tangent* to V. Then the integral curves of X^{L} contained in V are solutions of the Euler–Lagrange equation iff X^{L} satisfies

$$\mathrm{T}(\mathcal{F}L)\circ X^{\mathrm{L}} \underset{V}{\approx} K \tag{7.1}$$

(the weak equality means equality on the points of the submanifold V).

As a first approximation to this problem, let us call V_1 the subset of points $u \in \mathrm{T}Q$ where the linear equation—for the unknown vector a_u—$\mathrm{T}_u(\mathcal{F}L)\cdot a_u = K(u)$ is consistent, and assume it to be a submanifold, the primary Lagrangian constraint submanifold. Then the equation

$$\mathrm{T}(\mathcal{F}L)\circ X^{\mathrm{L}} \underset{V_1}{\approx} K \tag{7.2}$$

has solutions, let us call them *primary dynamical vector fields* [GP 92a]. They are not unique on V_1, since they can be added vector fields in $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$. On the other hand, one should find solutions that are tangent to V_1, and this is the beginning of an algorithm that, under some regularity conditions, may give at the end all the solutions of the Euler–Lagrange equation. This is like the Dirac theory in the Lagrangian formalism (see a careful discussion in [GP 92a]; see also [BGPR 86, MR 92]).

Note that any integral curve of a primary dynamical field X^{L} which is *contained* in V_1 is a solution of the Euler–Lagrange equation.

Our purpose now is to show that the choice of the Hamiltonian function H and the set of primary Hamiltonian constraints ϕ_μ yields a primary dynamical field X^{L}. Let us define the vector field

$$X^{\mathrm{L}}_o = \Delta_H + v^\mu \Delta_\mu. \tag{7.3}$$

Theorem 3. *The vector field* X_o^{L} *satisfies the second-order condition, and is a primary dynamical field. More precisely,*

$$\mathrm{T}(\mathcal{F}L) \circ X_o^{\mathrm{L}} = K - \chi_\mu \Upsilon^{v^\mu} \underset{V_1}{\approx} K. \tag{7.4}$$

Proof. A second-order vector field on $\mathrm{T}Q$ can be characterized by the property that $\mathrm{J} \circ X = \Delta_{\mathrm{T}Q}$. We have

$$\mathrm{J} \circ (\Delta_H + v^\mu \Delta_\mu) = \Gamma_H + v^\mu \Gamma_\mu = \Delta_{\mathrm{T}Q}$$

by (5.14) and (4.6), so X_o^{L} satisfies the second-order condition.

Now let us apply $\mathrm{T}(\mathcal{F}L)$ to X_o^{L}, and use (5.7):

$$\mathrm{T}(\mathcal{F}L) \circ X_o^{\mathrm{L}} = Z_H \circ \mathcal{F}L + v^\mu Z_\mu \circ \mathcal{F}L + \left(\mathcal{F}L^*\{H, \phi_\mu\} + v^\nu\, \mathcal{F}L^*\{\phi_\nu, \phi_\mu\}\right) \Upsilon^{v^\mu}.$$

In this expression we recognize the operator K (see equation (4.18)) and the primary Lagrangian constraints $\chi_\mu = K \cdot \phi_\mu$, thus obtaining (7.4). □

Before proceeding it will be interesting to note some additional properties of X_o^{L}. (We will use the notation $Y_\mu = Y_{\phi_\mu}$ and $R_\mu = R_{\phi_\mu}$.)

Proposition 5. *The vector field* X_o^{L} *satisfies the following properties:*

$$X_o^{\mathrm{L}} = Y_H + v^\mu Y_\mu \tag{7.5}$$

$$X_o^{\mathrm{L}} \cdot \mathcal{F}L^*(h) = K \cdot h - \chi_\mu\, \Gamma_h \cdot v^\mu \tag{7.6}$$

$$X_o^{\mathrm{L}} \cdot v^\nu = \chi_\mu\, M(\mathcal{F}v^\nu, \mathcal{F}v^\mu) \underset{V_1}{\approx} 0 \tag{7.7}$$

$$X_o^{\mathrm{L}} \cdot (K \cdot h) = K \cdot \{h, H\} + v^\mu K \cdot \{h, \phi_\mu\} + \chi_\nu \left(-R_h \cdot v^\nu + \mathcal{F}L^*\{h, \phi_\mu\} M(\mathcal{F}v^\mu, \mathcal{F}v^\nu)\right). \tag{7.8}$$

Proof. The first statement is an immediate consequence of the definition of X_o^{L} and the fact that

$$R_H + v^\nu R_\nu = 0 \tag{7.9}$$

whose proof is $R_H + v^\nu R_\nu = -v^\mu \Gamma_{\{\phi_\mu, H\}} + v^\nu \left(\Gamma_{\{\phi_\nu, H\}} + v^\mu \Gamma_{\{\phi_\nu, \phi_\mu\}}\right) = \Gamma_{\{\phi_\nu, \phi_\mu\}} v^\nu v^\mu = 0$, due to the antisymmetry of $\{\phi_\nu, \phi_\mu\}$.

The second one is a direct consequence of equation (7.4): it tells us the action of X_o^{L} (and indeed of any primary dynamical field X^{L}) on projectable functions.

The third equation gives the action of X_o^{L} on the non-projectable functions v^μ. It is obtained from (5.15) and the definition of the primary Lagrangian constraints χ_μ:

$$\begin{aligned} X_o^{\mathrm{L}} \cdot v^\nu &= (\Delta_H + v^\mu \Delta_\mu) \cdot v^\nu = \left(\mathcal{F}L^*\{\phi_\mu, H\} + v^\rho \mathcal{F}L^*\{\phi_\mu, \phi_\rho\}\right) M(\mathcal{F}v^\nu, \mathcal{F}v^\mu) \\ &= K \cdot \phi_\mu\, M(\mathcal{F}v^\nu, \mathcal{F}v^\mu) = \chi_\mu\, M(\mathcal{F}v^\nu, \mathcal{F}v^\mu). \end{aligned}$$

The fourth equation is obtained from $K \cdot h = \mathcal{F}L^*\{h, H\} + \sum_\mu \mathcal{F}L^*\{h, \phi_\mu\}\, v^\mu$, (4.18), by applying (7.6) and (7.7). □

As a consequence of the theorem we obtain the general form of a primary dynamical field in Lagrangian formalism:

$$X^{\mathrm{L}} = X_o^{\mathrm{L}} + \varepsilon^\mu\, \Gamma_\mu.$$

On the other hand, according to (4.13), the primary dynamical fields in the Hamiltonian formalism are

$$X^{\mathrm{H}} = Z_H + \lambda^\mu\, Z_\mu.$$

Both vector fields exhibit a set of arbitrary functions, ε^μ on $\mathrm{T}Q$ and λ^μ on T^*Q, and we can relate the corresponding dynamics.

Proposition 6. *Let $\xi\colon I \to \mathrm{T}Q$, $\eta\colon I \to \mathrm{T}^*Q$ be related solutions of the Euler–Lagrange and Hamilton–Dirac equations corresponding to the dynamical vector fields*

$$X^{\mathrm{L}} = X^{\mathrm{L}}_o + \varepsilon^\mu\, \Gamma_\mu \qquad X^{\mathrm{H}} = Z_H + \lambda^\mu\, Z_\mu.$$

Then the 'arbitrary functions' ε^μ, λ^μ are related by

$$\lambda^\mu(\eta(t)) = v^\mu(\xi(t)) \tag{7.10}$$

$$\varepsilon^\mu(\xi(t)) = (K \cdot \lambda^\mu)(\xi(t)). \tag{7.11}$$

Proof. We have

$$\dot\eta = Z_H \circ \eta + (\lambda^\mu \circ \eta)\, Z_\mu \circ \eta.$$

Since ξ and η are related, application of $\mathrm{T}(\tau^*_Q)$ yields

$$\xi = \mathcal{F}H \circ \eta + (\lambda^\mu \circ \eta)\, \mathcal{F}\phi_\mu \circ \eta = \mathcal{F}H \circ \mathcal{F}L \circ \xi + (\lambda^\mu \circ \eta)\, \mathcal{F}\phi_\mu \circ \mathcal{F}L \circ \xi$$

and from (4.3)

$$\xi = \gamma_H \circ \xi + (v^\mu \circ \xi)\, \gamma_\mu \circ \xi;$$

comparing both expressions we identify λ^μ with v^μ.

Now we compute

$$\begin{aligned}(K \cdot \lambda^\mu)(\xi(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\lambda^\mu(\eta(t)) = \frac{\mathrm{d}}{\mathrm{d}t} v^\mu(\xi(t)) \\ &= X^{\mathrm{L}} \cdot v^\mu = (X^{\mathrm{L}}_o + \varepsilon^\nu\, \Gamma_\nu) \cdot v^\mu \\ &= \varepsilon^\mu(\xi(t))\end{aligned}$$

where we have used (7.10) and the properties $X^{\mathrm{L}}_o \cdot v^\mu \underset{V_1}{\approx} 0$, $\Gamma_\nu \cdot v^\mu = \delta^\mu_\nu$. □

Another application of the properties of X^{L}_o is the relation between the Lagrangian and the Hamiltonian stabilization algorithms. For instance, putting $\phi^1_\mu = \{\phi_\mu, H\}$ (this is a secondary Hamiltonian constraint when ϕ_μ is first class) from (7.8) we have

$$X^{\mathrm{L}}_o \cdot (K \cdot \phi_\rho) = K \cdot \phi^1_\rho + v^\mu K \cdot \{\phi_\rho, \phi_\mu\} + \chi_\nu \left(-R_\rho \cdot v^\nu + \mathcal{F}L^*\{\phi_\rho, \phi_\mu\} M(\mathcal{F}v^\mu, \mathcal{F}v^\nu)\right)$$

and so for first-class constraints we obtain

$$X^{\mathrm{L}}_o \cdot (K \cdot \phi_{\mu_o}) \underset{V_1}{\approx} K \cdot \phi^1_{\mu_o}$$

which means that performing the first step of the Hamiltonian stabilization followed by application of K is equivalent to applying K and then performing the first step of the Lagrangian stabilization.

In a similar way from (7.6) we obtain

$$X^{\mathrm{L}}_o \cdot \mathcal{F}L^* \phi^1_{\mu_o} \underset{V_1}{\approx} K \cdot \phi^1_{\mu_o}.$$

In [BGPR 86] a vector field similar to the dynamical vector field X_o^{L} was introduced in coordinates, and was used in [Pon 88] to explore the relations between Lagrangian and Hamiltonian dynamics for singular Lagrangians. However, the simplest way to relate both dynamics is achieved with the choice of X_o^{L}.

On the other hand, in [Grà 00] an intrinsic way to construct a primary dynamical field in the Lagrangian formalism out from any second-order vector field was introduced using the Euler–Lagrange operator $\mathcal{E}_L$ and the map M given by equation (4.5). This procedure, when applied to the primary dynamical fields, leaves them invariant 'on-shell' (we mean on the primary Lagrangian constraint submanifold). The vector field X_o^{L} is special among the primary dynamical fields in the sense that its action on the non-projectable functions v^μ is zero on-shell.

Canonical symmetries and canonical Noether symmetries. Now we shall re-express some statements about symmetries using the vector field Y_h.

Let us consider the time-independent symmetries in phase space that are generated by a function G on phase space through the Hamiltonian vector field $Z_G = \{-, G\}$. It turns out [GP 88] that the necessary and sufficient condition for a function G to generate in this way an infinitesimal symmetry of the Hamilton–Dirac equation of motion is that

$$K \cdot G \underset{V_f}{\simeq} c \tag{7.12}$$

for some constant c (in the time-dependent case this would be a function $c(t)$). Here $\simeq$ denotes Dirac's strong equality, that is, an equality up to quadratic terms in the constraints—now the whole set of constraints, corresponding to the final Lagrangian constraint submanifold V_f [BGPR 86, GP 92a].

Then, application of (5.6) yields

$$Y_G \cdot (K \cdot h) \underset{V_f}{\approx} K \cdot \{h, G\} \tag{7.13}$$

for every function h, where $\approx$ means equality over the whole constraint surface.

Note conversely that if a function G satisfies (7.13) for every function h, then (5.6) implies that $Y_h \cdot (K \cdot G) \underset{V_f}{\approx} 0$ for each h, and so we obtain (7.12) again. We have thus obtained the following:

Theorem 4. *The necessary and sufficient condition for the Hamiltonian vector field Z_G to generate a symmetry of the Hamilton–Dirac equation of motion is*

$$Y_G \cdot (K \cdot h) \underset{V_f}{\approx} K \cdot (Z_G \cdot h) \tag{7.14}$$

for all functions h.

One can also consider the more restrictive case of canonical Noether symmetries, whose infinitesimal generator G can be characterized in a similar way [BGGP 89] as

$$K \cdot G = c. \tag{7.15}$$

Then the same reasoning as above leads to the following:

Theorem 5. *The necessary and sufficient condition for the Hamiltonian vector field Z_G to generate a Noether symmetry in phase space is that*

$$Y_G \cdot (K \cdot h) = K \cdot (Z_G \cdot h) \tag{7.16}$$

for all functions h.

Note the remarkable fact that a weak (on-shell) equality or a standard equality is the only difference between the characterization (7.14) for a symmetry of the Hamilton–Dirac equation of motion and the characterization (7.16) for a canonical Noether symmetry. Since Noether symmetries exhibit a property of the action functional, it is clear that their characterization must be, as we see, on- and off-shell. This characterization (7.16) was first obtained in the paper [GP 00], which was instrumental in finding the new geometric structures that have been introduced in the present paper.

Note also that, when $c \neq 0$ in (7.12) or (7.15), the conserved quantity associated with the symmetry is $G - ct$ rather than G.

8. The case of a regular Lagrangian

In this section we will show what the preceding results become when the Lagrangian is hyperregular, namely, when $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ is a diffeomorphism—in a local study, we might suppose only that the Lagrangian is regular, namely, that $\mathcal{F}L$ is a local diffeomorphism.

Now the 2-form $\omega_L = \mathcal{F}L^*(\omega_Q)$ on $\mathrm{T}Q$ is symplectic. Let us denote by X_f the Hamiltonian vector field of a function f with respect to ω_L. Recall that the Lagrangian dynamics is now ruled by the Hamiltonian vector field $X^{\mathrm{L}} = X_{E_L}$ of the energy function.

Proposition 7. *Suppose that the Lagrangian is hyperregular. Then*

$$\Gamma_h = \mathrm{J} \circ X_{\mathcal{F}L^*(h)} \tag{8.1}$$

$$R_h = \mathrm{J} \circ X_{\mathcal{F}L^*\{h,H\}} \tag{8.2}$$

$$\Delta_h = X_{\mathcal{F}L^*h} \tag{8.3}$$

$$Y_h = X_{\mathcal{F}L^*(h)} + \mathrm{J} \circ X_{\mathcal{F}L^*\{h,H\}}. \tag{8.4}$$

Proof. The vertical vector fields in (8.1) correspond to bundle maps $\mathrm{T}Q \to \mathrm{T}Q$. For the right-hand side the map is

$$\mathrm{T}(\tau_Q) \circ X_{\mathcal{F}L^*(h)} = \mathrm{T}(\tau_Q) \circ \mathrm{T}(\mathcal{F}L^{-1}) \circ Z_h \circ \mathcal{F}L = \mathrm{T}(\tau_Q^*) \circ Z_h \circ \mathcal{F}L$$

which coincides with the map $\gamma_h = \mathcal{F}h \circ \mathcal{F}L$ that corresponds to Γ_h.

Definition (5.8) when there are no constraints yields $R_h = \Gamma_{\{h,H\}}$. Then equation (8.2) follows immediately from (8.1). (Note by the way that $R_H = 0$.)

Another consequence of the non-existence of constraints is that, according to (5.17) or theorem 1, Δ_h projects to the Hamiltonian vector field Z_h, and thus it is the Hamiltonian vector field of $\mathcal{F}L^*(h)$, which is the content of (8.3).

Finally, the last equation is an immediate consequence of the definition $\Delta_h = Y_h - R_h$. □

Given a second-order vector field D on $\mathrm{T}Q$, a vector field X is called *newtonoid* with respect to D (see, for instance, [MM 86, CLM 89] and references therein) if $\mathrm{J} \circ [X, D] = 0$. From any vector field X one can construct a newtonoid vector field—with respect to D—as $X + \mathrm{J} \circ [D, X]$. This construction, which has been used in several papers to study the symmetries of Lagrangian dynamics, is a kind of generalization of the complete lift of a vector field on Q to $\mathrm{T}Q$. From equation (8.4) it is then easy to deduce the following result:

Corollary 1. *If the Lagrangian is hyperregular then Y_h is a newtonoid vector field with respect to the dynamical vector field X_o^{L} of velocity space, and is the newtonoid vector field defined from the vector field $X_{\mathcal{F}L^*(h)} = \Delta_h$.*

In the singular case, using (7.6) it is readily seen that Y_h satisfies the condition of being newtonoid with respect to X_o^{L} only on the primary Lagrangian constraint submanifold V_1.

9. An example

As a simple example, let us consider the Lagrangian of the conformal particle [Sie 88, GR 93]

$$L = \tfrac{1}{2}(\dot{x}^2 - \lambda x^2) \tag{9.1}$$

with configuration variables $(x, \lambda) \in Q = \mathbb{R}^n \times \mathbb{R}$, and $\mathbb{R}^n$ endowed with an indefinite scalar product. The Legendre transformation is given by

$$\mathcal{F}L(x, \lambda; \dot{x}, \dot{\lambda}) = (x, \lambda; \hat{p}, \hat{\pi}) \qquad \hat{p} = \dot{x}, \ \hat{\pi} = 0 \tag{9.2}$$

so the primary constraint submanifold $P_o \subset \mathrm{T}^*Q$ has codimension one, and is described by the primary Hamiltonian constraint

$$\phi = \pi. \tag{9.3}$$

As a Hamiltonian we take

$$H = \tfrac{1}{2}(p^2 + \lambda x^2). \tag{9.4}$$

Stabilization of $\phi^0 = \phi$ yields three additional generations of constraints $\phi^{i+1} = \{\phi^i, H\}$:

$$\phi^1 = -\tfrac{1}{2}x^2 \qquad \phi^2 = -px \qquad \phi^3 = \lambda x^2 - p^2$$

which are first class. The Lagrangian constraints are $\chi^i := K \cdot \phi^{i-1}$:

$$\chi = \chi^1 = -\tfrac{1}{2}x^2 \qquad \chi^2 = -\dot{x}x \qquad \chi^3 = \lambda x^2 - \dot{x}^2.$$

(Indeed, $\chi^i = \mathcal{F}L^*(\phi^i)$, since the Hamiltonian constraints are first class.) Note also that $K \cdot \phi^3 = -2\dot{\lambda}\chi^1 - 4\lambda\chi^2$.

The kernel of $\mathrm{T}(\mathcal{F}L)$ is spanned by $\Gamma_\phi = \partial/\partial\dot{\lambda}$. From the identity $\mathrm{Id} = \gamma_H + v\,\gamma_\phi$ we determine the function $v = \dot{\lambda}$. We also obtain

$$K \cdot g = \dot{x}^a\,\mathcal{F}L^*\left(\frac{\partial g}{\partial x^a}\right) + \dot{\lambda}\,\mathcal{F}L^*\left(\frac{\partial g}{\partial \lambda}\right) - \lambda x_a\,\mathcal{F}L^*\left(\frac{\partial g}{\partial p_a}\right) - \frac{1}{2}x^2\,\mathcal{F}L^*\left(\frac{\partial g}{\partial \pi}\right)$$
$$= \mathcal{F}L^*\{g, H\} + \mathcal{F}L^*\{g, \pi\}\,\dot{\lambda}.$$

Now we can compute

$$Y_h = \mathcal{F}L^*\left(\frac{\partial h}{\partial p}\right)\frac{\partial}{\partial x} + \mathcal{F}L^*\left(\frac{\partial h}{\partial \pi}\right)\frac{\partial}{\partial \lambda} + \left(K \cdot \frac{\partial h}{\partial p}\right)\frac{\partial}{\partial \dot{x}} + \left(K \cdot \frac{\partial h}{\partial \pi}\right)\frac{\partial}{\partial \dot{\lambda}}$$

and, in particular,

$$Y_\phi = \frac{\partial}{\partial \lambda} \qquad Y_H = \dot{x}\frac{\partial}{\partial x} - \lambda x\frac{\partial}{\partial \dot{x}}.$$

Then, from $R_h = \Gamma_{\{h,H\}} + \dot{\lambda}\,\Gamma_{\{h,\pi\}}$ we obtain $R_\phi = \Gamma_{\phi^1} = 0$ and $R_H = \dot{\lambda}\,\Gamma_{-\phi^1} = 0$, from which $\Delta_\phi = Y_\phi$ and $\Delta_H = Y_H$.

According to our results, the kernel of the presymplectic form ω_L is spanned by $\Gamma_\phi = \partial/\partial\dot{\lambda}$ and $\Delta_\phi = \partial/\partial\lambda$. (In this case this is obvious since $\omega_L = \mathrm{d}x \wedge \mathrm{d}\dot{x}$.)

Finally, we obtain the primary dynamical vector fields as $X^{\mathrm{L}} = X^{\mathrm{L}}_o + \varepsilon\Gamma_\phi$, where

$$X^{\mathrm{L}}_o = Y_H + \dot{\lambda}\,Y_\phi = \dot{x}\frac{\partial}{\partial x} + \dot{\lambda}\frac{\partial}{\partial \lambda} - \lambda x\frac{\partial}{\partial \dot{x}}.$$

It is easily checked that

$$\mathrm{T}(\mathcal{F}L) \circ X^{\mathrm{L}}_o - K = -\chi\,\frac{\partial}{\partial \pi} \approx 0.$$

10. Conclusions

During the previous two decades many papers have studied the close relations between Lagrangian and Hamiltonian formalisms when the Lagrangian function is singular. One can expedite the Lagrangian picture by using some results from the Hamiltonian side.

In this paper we have added new objects to the geometric framework of these relations. First, for any function h on phase space T^*Q we have defined the vector field Y_h on velocity space TQ. When viewed in coordinates, this object reminds one of the definition of newtonoid vector fields; but instead of using a second-order dynamics on Q, which is not well defined in general when the Lagrangian is singular, we use the unambiguous time-evolution operator K that connects Lagrangian and Hamiltonian formalisms. Once a Hamiltonian H and a set of primary Hamiltonian constraints ϕ_μ have been chosen, we have also defined the vector fields R_h and Δ_h.

These objects give effective answers to several questions. The projectability of a vector field to a Hamiltonian vector field: we have shown that, when h is a first-class function on T^*Q, the vector field Δ_h projects to the Hamiltonian vector field Z_h. The kernel of the presymplectic form of the Lagrangian formalism: it can be computed as the subbundle spanned by the vector fields Γ_μ associated with the primary Hamiltonian constraints ϕ_μ and the vector fields Δ_{μ_o} associated with the first-class primary Hamiltonian constraints. The construction of the dynamical vector fields in the Lagrangian formalism: the vector field $X_o^{\mathrm{L}} = \Delta_H + v^\mu \Delta_\mu$ is a solution of the Euler–Lagrange equation on the primary Lagrangian constraint submanifold. Finally, the characterization of dynamical symmetries: the fact that G is the generator of an infinitesimal symmetry can be expressed as a kind of commutation relation between the time-evolution operator K and the couple of vector fields Y_G, Z_G.

In view of these results, we can say that the time-evolution operator K still provides one with new insights concerning the connections between singular Lagrangian and Hamiltonian dynamics. The functions v^μ, given by (4.3) as a kind of pseudo-inversion of the Legendre transformation, and the fibre derivation, a seldom used operation in geometric mechanics, complete, together with the usual structures of tangent and cotangent bundles, the set of tools used in this paper.

As a final remark, let us point out that some of our expressions are also valid in the time-dependent case, which is especially interesting for dealing with gauge symmetries.

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